'Mirror model' gives separation of convolutive mixing of PNL mixtures

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The proof is given that the so called 'mirror model' as demixing model is able to recover original sources after non-trivial mixing. The issue explored is the capability to separate sources, in a blind way, after the convolutive mixing of post nonlinear (PNL) mixtures. The strictness of that kind of mixture produces non-trivial problems in separating signals without any adequate assumption on recovering architecture.

Introduction: It is widely reported in the literature that nonlinear mixing is a non-trivial problem in the ICA framework. The independence conservation constraint is not strong enough to recover original sources from nonlinear mixtures if they have no particular form or there are no other assumptions about the mixing (and demixing) model [1]. In non-trivial mixing environments the independent solution performed with blind source separation can not be the desired one. If no *a priori* assumption is given about the mixing–demixing model, there exist some maps able to produce independent output with non-diagonal Jacobian matrix; this kind of map preserves the output independence but does not recover the original signals. In general, the ICA approach on blind source separation could not achieve the desired result, so adding some 'soft' constraint to the problem produces the uniqueness of the solution (the solution has to be the desired one at least of trivial indeterminacies [2]). This implies that the main issue is to ensure the presence of conditions (in terms of sources, mixing environment, recovering structure) granting at least theoretically the possibility of achieving the desired solution. The mixing model is not a free parameter; it depends on the mixing environment. In this Letter, we consider environments that are modelled as in Fig. 1, and written in a close form as:

$$\boldsymbol{x}[n] = \sum_{k=0}^{L-1} \boldsymbol{Z}[k]\boldsymbol{F}[\boldsymbol{A}\boldsymbol{s}[n-k]] \qquad (1)$$

in which $\boldsymbol{s}[n]$ is the $N \times 1$ vector of source signals, $\boldsymbol{A}$ is an $N \times N$ static matrix, $\boldsymbol{F}[\boldsymbol{r}(n)] = [f_1[r_1(n)], \ldots, f_N[r_N(n)]]^T$ is the $N \times 1$ vector of nonlinear distorting functions, one for each channel, and

$$\boldsymbol{Z}[k] = \begin{bmatrix} z_{11}[k] & & z_{1N}[k] \\ & \ddots & \\ z_{N1}[k] & & z_{NN}[k] \end{bmatrix}$$

is an FIR matrix, in which each element is an L-tap FIR filter. This Letter shows that the uniqueness of a BSS solution, after the novel mixing environment here proposed, is granted if and only if the demixing structure is modelled on the base of the so-called 'mirror model' (the proof is an extension of 'lemma 1' introduce by Taleb in [3] about the PNL model). If the mixing environment is modelled as in (1) the mirror model is expressed in a closed form as follows:

$$\boldsymbol{y}[n] = \boldsymbol{B}\boldsymbol{G}\left[\sum_{k=0}^{L-1} \boldsymbol{W}[k]\boldsymbol{x}[n-k]\right] \qquad (2)$$

For an overview on the mixing demixing structure, see Fig. 2.

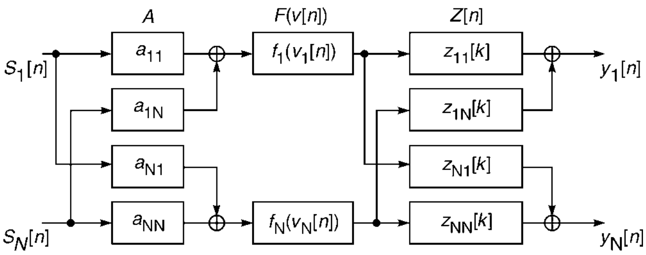

Fig. 1 *Hidden mixing model*

Proposition: Given a convolutive, nonlinear mixture model F: $\{\boldsymbol{A}, \boldsymbol{F}, \boldsymbol{Z}\}$ (1) and considering the 'mirror model' G: $\{\boldsymbol{B}, \boldsymbol{G}, \boldsymbol{W}\}$ as a recovery model (2); assuming that:

(*a*) the static mixing matrix $\boldsymbol{A}$ is non-singular with nonzero elements
(*b*) $\forall i = 1, \ldots, N$ g_i, f_i are differentiable, invertible, monotone and $g_i[0] = f_i[0] = 0$
(*c*) $\boldsymbol{s}[n]$ is a random vector in which components are spatially independent and temporally white.
(*d*) the PDF of $\boldsymbol{s}[n]$ vanishes for at least one real $\boldsymbol{s}$.

The output $\boldsymbol{y}[n]$ is a vector composed of a spatially independent temporally white component if and only if the output can be expressed as:

$$\boldsymbol{y}[n] = \boldsymbol{P}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}\begin{bmatrix} z^d & & 0 \\ & \ddots & \\ 0 & & z^d \end{bmatrix}\boldsymbol{s}[n] \qquad (3)$$

in which $\boldsymbol{P}$ is a permutation matrix and $\lambda_1, \ldots \lambda_N$ are scaling factors.

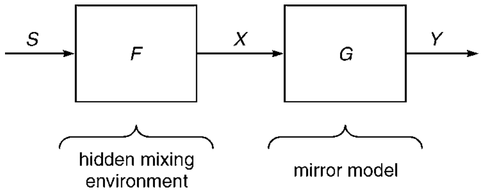

Fig. 2 *Mixing–demixing structure*

Proof: Sufficient condition ($\Leftarrow$) (existence condition): If the in–out model is (3), it is trivial that for $\boldsymbol{s}[n]$ white and spatially independent random vector, $\boldsymbol{y}[n]$ must be spatially independent too. This channel does not produce any mixing between different components and does not introduce component correlations.

Necessary condition ($\Rightarrow$) (uniqueness condition): If $\boldsymbol{y}[n]$ is a spatially independent random vector the channel model must be of the kind (3). The output signals expressed with respect to hidden sources are:

$$\begin{aligned}\boldsymbol{y}[n] &= \boldsymbol{B}\boldsymbol{G}\left[\sum_{k=0}^{L-1} \boldsymbol{W}[k] \sum_{k=0}^{L-1} \boldsymbol{Z}[k]\boldsymbol{F}\{\boldsymbol{A}\boldsymbol{s}[n-k]\}\right] \\ &= \boldsymbol{B}\boldsymbol{G}[\boldsymbol{H}[n] * \boldsymbol{F}[\boldsymbol{A}\boldsymbol{s}[n]]] \end{aligned} \qquad (4)$$

Then two propositions have to be introduced:

Proposition A: Considering the channel $h_{ij}[k]$: $\boldsymbol{H}[k] = \boldsymbol{W}[k]*\boldsymbol{Z}[k]$ of (4), let $\boldsymbol{D}$ be the set $\boldsymbol{D} = \{(i, j) | \ \exists \bar{k}\text{: } h_{i,j}\ [\bar{k}] \neq 0\}$. For each possible couple $(i, j) \in \boldsymbol{D}$ the values $k = \bar{k}$ for which $h_{i,j}[\bar{k}] \neq 0$ are no more than one. Moreover $\forall (i, j) \neq (p, q) \in \boldsymbol{D} \Rightarrow \bar{k}_{ij} = \bar{k}_{pq}$.

Proof of proposition A: If the number of k was more than one, considering the mirror model as demixing model, the output signal $\boldsymbol{y}[n]$ could not be white as the hypothesis assumes. If different couples (i, j) and (k, l) of $\boldsymbol{D}$ should have $k_{ij} \neq k_{kl}$, the recovery structure (2) could not be able any more to provide independent outputs.

Proposition B: For a given $k = \bar{k}$ accepting the results from proposition A, in each row (or column) of matrix $\boldsymbol{H}$ there are no pairs $(i, j) \in \boldsymbol{D}$ having the same corresponding row or column index, moreover the cardinality of D cannot be smaller than the matrix rank, i.e. (i, j), $(h, k) \in \boldsymbol{D} \Rightarrow i \neq h$ and $j \neq k$.

Proof of proposition B: If the cardinality of $\boldsymbol{D}$ was less than the matrix rank, it could not be possible any more to obtain all the independent output components. Moreover, if more than one couple (i, j) of the $\boldsymbol{D}$ set had the same first (or second) row or column index, at least two output variables could not be independent any more, considering the recovering structure (2). Considering the results of both propositions A and B, it is possible to express the PDF of the vector $\boldsymbol{s}$ as a function of the output PDF because the problem can now be approached as a static problem:

$$\begin{aligned} p_s(\boldsymbol{s}) &= \prod_{i=1}^{N} p_{s_i}(s_i) \\ &= \prod_{i=1}^{N} p_{y_i}\left(\sum_{j=1}^{N} b_{ij} g_j\left[\sum_{q=1}^{N}\sum_{k=0}^{L-1} h_{jq}[k] f_q\left(\sum_{m=1}^{N} a_{qm} s_m[n-k]\right)\right]\right) \\ &\quad \times |\boldsymbol{J}| \quad \forall \boldsymbol{s} \in R^N \end{aligned} \qquad (5)$$

From hypothesis (*d*) $\exists \bar{s} \in \boldsymbol{R}^N$ so that $p_s(\bar{\boldsymbol{s}}) \equiv 0$. From (5), for a non-null Jacobian J, there exists at least some $\boldsymbol{y}^0 = [y_1^0, \ldots, y_N^0] \in \boldsymbol{R}^N$ such that $\prod_{i=1}^{N} p_{y_i}(y_i) = 0$. So there exists an integer i such that $p_{y_i}(y_i^0) = 0$, the value of y_i for which the marginal PDF is null can be expressed as follows:

$$y_i^0 = \sum_{j=1}^{N} b_{ij} g_j \left[\sum_{q=1}^{N} \sum_{k=0}^{L-1} h_{jq}[k] f_q \left(\sum_{m=1}^{N} a_{qm} s_m[n-k] \right) \right] \quad (6)$$

The previous relations are valid for all $i = 1, \ldots, N$. Solutions of the implicit equation (6) lie on $H_i(\boldsymbol{s})$: hypersurface of $\boldsymbol{R}^N$. It is evident that $H_i(\boldsymbol{s}) = Ker[p_s(\boldsymbol{s})]$. For a given i, $H_i(\boldsymbol{s})$ is parallel to the hyperplane orthogonal to the axis s_i. Suppose that $H_i(\boldsymbol{s})$ is not parallel to any $s_i = 0$ planes: the projection of $H_i(\boldsymbol{s})$ on s_i should be equal to R. $\forall s_i \in \boldsymbol{R}$ $\exists s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N$: $\boldsymbol{s} \in \boldsymbol{H}_i \Rightarrow p_s(\boldsymbol{s}) \equiv 0$. This cannot be true because $\int_S p_s(\boldsymbol{s}) ds = 1$. Considering the results of propositions A and B, without any loss of generality, it should be noted that:

$$\sum_{j=1}^{N} b_{ij} g_j \left[h_{jq}[k] f_q \left(\sum_{m=1}^{N} a_{qm} s_m[n-k] \right) \right] = w_{\sigma(i)}\left(s_{\sigma(i)}\right) \quad i = 1 \ldots N \quad (7)$$

where $w_{\sigma(i)}(s_{\sigma(i)})$ is a generic function depending only by $s_{\sigma(i)}$. Equations (6) and (7) can be expressed for each channel. Deriving (7) with respect to $\boldsymbol{s}$, without any loss of generality taking $\sigma(i) = i$, it results that:

$$\begin{bmatrix} \dot{w}_1(s_1) & & 0 \\ & \ddots & \\ 0 & & \dot{w}_N(s_i) \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} \dot{g}_1 \left[h_{1q}[k] f_q \left(\sum_{m=1}^{N} a_{qm} s_m[n-k] \right) \right] & & 0 \\ & \ddots & \\ 0 & & \dot{g}_N[\boldsymbol{s}] \end{bmatrix} \times \begin{bmatrix} h_{11}[k_1] & & 0 \\ & \ddots & \\ 0 & & h_{NN}[k_N] \end{bmatrix} \times \begin{bmatrix} \dot{f}_1 \left(\sum_{m=1}^{N} a_{1m} s_m[n-k] \right) & & 0 \\ & \ddots & \\ 0 & & \dot{f}_1(\boldsymbol{s}) \end{bmatrix} \boldsymbol{A} \quad (8)$$

Considering $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ as different elements of $H(\boldsymbol{s})$, (8) can be evaluated in $\boldsymbol{s}_1$ and in $\boldsymbol{s}_2$, as follows:

$$\begin{cases} \boldsymbol{D}(\boldsymbol{s}_1) = \boldsymbol{B} \boldsymbol{\Lambda}_{\dot{\mathbf{G}}}(\boldsymbol{s}_1) \boldsymbol{H} \boldsymbol{\Lambda}_{\dot{\mathbf{F}}}(\boldsymbol{s}_1) \boldsymbol{A} \\ \boldsymbol{D}(\boldsymbol{s}_2) = \boldsymbol{B} \boldsymbol{\Lambda}_{\dot{\mathbf{G}}}(\boldsymbol{s}_2) \boldsymbol{H} \boldsymbol{\Lambda}_{\dot{\mathbf{F}}}(\boldsymbol{s}_2) \boldsymbol{A} \end{cases} \rightarrow \begin{cases} \boldsymbol{D}(\boldsymbol{s}_1) = \boldsymbol{B} \boldsymbol{\Lambda}_{\dot{\mathbf{G}}\mathbf{H}\dot{\mathbf{F}}}(\boldsymbol{s}_1) \boldsymbol{A} \\ \boldsymbol{D}(\boldsymbol{s}_2) = \boldsymbol{B} \boldsymbol{\Lambda}_{\dot{\mathbf{G}}\mathbf{H}\dot{\mathbf{F}}}(\boldsymbol{s}_2) \boldsymbol{A} \end{cases} \quad (9)$$

in which $\boldsymbol{D}(\cdot)$ is a diagonal matrix. Eliminating $\boldsymbol{B}$ from (8):

$$\boldsymbol{A} \underbrace{\begin{bmatrix} d_{11}(\boldsymbol{s}_2, \boldsymbol{s}_1) & & 0 \\ & \ddots & \\ 0 & & d_{NN}(\boldsymbol{s}_2, \boldsymbol{s}_1) \end{bmatrix}}_{\boldsymbol{D}^{-1}(\boldsymbol{s}_2)\boldsymbol{D}(\boldsymbol{s}_1)} = \underbrace{\begin{bmatrix} \lambda_{11}(\boldsymbol{s}_2, \boldsymbol{s}_1) & & 0 \\ & \ddots & \\ 0 & & \lambda_{NN}(\boldsymbol{s}_2, \boldsymbol{s}_1) \end{bmatrix}}_{\boldsymbol{\Lambda}^{-1}_{\dot{G}H\dot{F}}(\boldsymbol{s}_2)\boldsymbol{\Lambda}_{\dot{G}H\dot{F}}(\boldsymbol{s}_1)} \boldsymbol{A}$$

As mentioned in hypothesis (a) the matrix $\boldsymbol{A}$ is regular. For each pair of nonzero elements of the matrix $\boldsymbol{A}$ it is possible to write:

$$\begin{cases} a_{ij}[d_{jj}(\boldsymbol{s}_2, \boldsymbol{s}_1) - \lambda_{ii}(\boldsymbol{s}_2, \boldsymbol{s}_1)] = 0 \\ a_{hj}[d_{jj}(\boldsymbol{s}_2, \boldsymbol{s}_1) - \lambda_{hh}(\boldsymbol{s}_2, \boldsymbol{s}_1)] = 0 \end{cases}$$
$$\Rightarrow \lambda_{ii}(\boldsymbol{s}_2, \boldsymbol{s}_1) = \lambda_{hh}(\boldsymbol{s}_2, \boldsymbol{s}_1) \quad \forall \boldsymbol{s}_2, \boldsymbol{s}_1 \in \boldsymbol{H} \quad (10)$$

From (10) it follows that:

$$\frac{\dot{g}_i[\tilde{f}_{\sigma(i)}((\boldsymbol{A})_{\sigma(i)}\boldsymbol{s}_1)]\dot{\tilde{f}}_{\sigma(i)}((\boldsymbol{A})_{\sigma(i)}\boldsymbol{s}_1)}{\dot{g}_h[\tilde{f}_{\sigma(h)}((\boldsymbol{A})_{\sigma(h)}\boldsymbol{s}_2)]\dot{\tilde{f}}_{\sigma(h)}((\boldsymbol{A})_{\sigma(h)}\boldsymbol{s}_2)} = C \quad \forall \boldsymbol{s}_2, \boldsymbol{s}_1 \quad (11)$$

in which $\tilde{f}_{\sigma(i)}((\boldsymbol{A})_{\sigma(i)}\boldsymbol{s}_1) = \alpha f_{\sigma(i)}((\boldsymbol{A})_{\sigma(i)}\boldsymbol{s}_1)$ and C is constant. The two linear forms that appear in (11), $(\boldsymbol{A})_{\sigma(i)}\boldsymbol{s}_1$ and $(\boldsymbol{A})_{\sigma(\tilde{i})}\boldsymbol{s}_1$, are independent as assumed in hypothesis (a). It is possible to express (11) in the following way: $\dot{g}_i[\tilde{f}_{\sigma(i)}(x)]\dot{\tilde{f}}_{\sigma(i)}(x) = C\dot{g}_{\tilde{i}}[\tilde{f}_{\sigma(\tilde{i})}(y)]\dot{\tilde{f}}_{\sigma(\tilde{i})}(y)$ $\forall x, y \in \boldsymbol{R}$; this can be true if and only if function $g_i(\cdot)$ is the inverse function of $f_{\sigma(i)}(\cdot)$ at least scaled for some scaling factor.

The method described above reduces the complex nonlinear convolutive problem (1) to a simpler static one for which the uniqueness of the solution is given at least for some trivial ambiguities that can be modelled with the factor of (3). Preliminary result of this approach is in [4], in which a feedforward blind adaptive network recovers the original signal from a convolutive mixing of PNL mixture.

2 December 2003

Electronics Letters online no: 20040243

doi: 10.1049/el:20040243

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